

On the Trace Anomaly and the Anomaly Puzzle in $\mathcal{N} = 1$ Pure Yang-Mills

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Abstract

The trace anomaly of the energy-momentum tensor is usually quoted in the form which is proportional to the beta function of the theory. However, there are in general many definitions of gauge couplings depending on renormalization schemes, and hence many beta functions. In particular, $\mathcal{N} = 1$ supersymmetric pure Yang-Mills has the holomorphic gauge coupling whose beta function is one-loop exact, and the canonical gauge coupling whose beta function is given by the Novikov-Shifman-Vainshtein-Zakharov beta function. In this paper, we study which beta function should appear in the trace anomaly in $\mathcal{N} = 1$ pure Yang-Mills. We calculate the trace anomaly by employing the $\mathcal{N} = 4$ regularization of $\mathcal{N} = 1$ pure Yang-Mills. It is shown that the trace anomaly is given by one-loop exact form if the composite operator appearing in the trace anomaly is renormalized in a preferred way. This result gives the simplest resolution to the anomaly puzzle in $\mathcal{N} = 1$ pure Yang-Mills. The most important point is to examine in which scheme the quantum action principle is valid, which is crucial in the derivation of the trace anomaly.

1 Introduction

In supersymmetric field theories, any operator is contained in some supermultiplet. This is also the case for the energy-momentum tensor $T^{\mu\nu}$. It was discovered long time ago [1] that the energy-momentum tensor is contained in a supermultiplet called the supercurrent \mathcal{J}^μ . The components of the supercurrent is given as

$$\mathcal{J}^\mu = j_R^\mu + \theta S^\mu + \text{h.c.} + 2\theta\sigma_\nu\bar{\theta}T^{\mu\nu} + \cdots, \quad (1)$$

where j_R^μ is an R-symmetry current and S^μ is the current of supersymmetry.

In the past, it was believed [2] that the conservation equation of the energy-momentum tensor is extended to the superspace equation given by

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} \approx D_\alpha X \quad (2)$$

where X is some chiral superfield, or else

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} \approx -\frac{1}{4}\bar{D}^2 D_\alpha J, \quad (3)$$

where J is some real vector superfield, and we use the symbol \approx for equations which are valid if equations of motion are used (i.e., the equations are valid up to contact terms when they are inserted into correlation functions). Eq. (3) is possible only if a conserved R-symmetry current exists, so we focus on Eq. (2) in the following.

Eq. (2) gives the conservation equations of $T^{\mu\nu}$ and S^μ , but it also gives the additional constraint given by

$$\frac{2}{3}T_\mu^\mu + i\partial_\mu j_R^\mu \approx F_X, \quad (4)$$

where F_X is the F -component of X . The fact that the right hand side of Eq. (4) is given by a single F -term causes a problem called the anomaly puzzle [3]. The axial anomaly contributing to $\partial_\mu j_R^\mu$ is often said to be one-loop exact due to the Adler-Bardeen theorem [4]. This one-loop exactness seems to indicate that the chiral superfield X is one-loop exact. On the other hand, the trace anomaly T_μ^μ usually receives higher order corrections. Then, higher order corrections to X seem to be necessary. These two statements look inconsistent with each other. This is the anomaly puzzle. Many works have been done to solve the anomaly puzzle, including Refs. [5, 6, 7, 8] and references therein.

The situation have been changed by the discovery of a new supercurrent multiplet [9]. (See also Ref. [10] for an early work.) It has been found that the supercurrent equation can be relaxed to the form

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} \approx D_{\alpha} X - \frac{1}{4} \bar{D}^2 D_{\alpha} J. \quad (5)$$

Then, Eq. (4) is now modified to

$$\frac{2}{3} T_{\mu}^{\mu} \approx \text{Re}(F_X) - \frac{1}{6} D_J, \quad (6)$$

$$\partial_{\mu} j_R^{\mu} \approx \text{Im}(F_X), \quad (7)$$

where D_J is the D -component of J . These equations suggest that X can be one-loop exact to maintain the one-loop exactness of the axial anomaly, and higher order corrections to the trace anomaly is accounted for by the new term D_J . Based on this observation, a resolution to the anomaly puzzle has been proposed [11]. (See also Ref. [12] for a related work.) More explicitly, let us consider a theory with matter chiral fields Φ_r which transform in the representation r of the gauge group. The gauge field strength chiral field is denoted as W_{α} , and the superpotential is denoted as W . Then, X and J are given as

$$X = \frac{4}{3} \left[3W - \sum_r \Phi_r \frac{\partial W}{\partial \Phi_r} - \frac{3t_{Ad} - \sum_r t_r}{32\pi^2} W^{\alpha} W_{\alpha} \right], \quad (8)$$

$$J = -4 \sum_r \gamma_r \Phi_r^{\dagger} e^{-2V} \Phi_r, \quad (9)$$

where t_{Ad} and t_r are the Dynkin indices for the adjoint representation and the representation r , respectively, γ_r is the matter anomalous dimension, and the summation is over all matter fields. Several consistency checks have been done for this proposal in Ref. [11].

However, there still remains a puzzle. The problem already exists in $\mathcal{N} = 1$ supersymmetric pure Yang-Mills, and we restrict our attention to this case. In this theory, there is no candidate for the operator J , and hence the situation regarding the anomaly puzzle of this theory have not been changed. The trace anomaly is often quoted in the form

$$T_{\mu}^{\mu} \approx -\frac{\beta}{4g^4} F_{\mu\nu} F^{\mu\nu} \quad (10)$$

where β is the beta function of the gauge coupling, and $F_{\mu\nu}$ is the gauge field strength. In supersymmetric theories, there is a known “exact” beta function [13, 7, 8], called the

Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function. In the case of the $\mathcal{N} = 1$ pure Yang-Mills, it is given as

$$\beta_{\text{NSVZ}} = \mu \frac{\partial g^2}{\partial \mu} = -\frac{1}{8\pi^2} \frac{3t_{Ad}g^4}{1 - t_{Ad}g^2/8\pi^2}. \quad (11)$$

If this beta function is inserted into Eq. (10), the trace anomaly is not one-loop exact. Then it is inconsistent with the Adler-Bardeen theorem and the supercurrent equation.

However, the beta function is obviously renormalization scheme dependent. For example, the beta function in the so-called DR (or $\overline{\text{DR}}$) scheme is not given by the NSVZ beta function beyond the two-loop level [14]. We can even define a gauge coupling, called the holomorphic gauge coupling, whose beta function is one-loop exact [7]. Furthermore, the composite operator $F_{\mu\nu}F^{\mu\nu}$ is also renormalization scheme dependent. Eq. (10) has been proven [15, 16, 17] by using the minimal subtraction scheme in dimensional regularization for QCD-like theories, but this scheme cannot preserve manifest supersymmetry. Because the anomaly puzzle originates from the superspace equation, we have to use a scheme in which supersymmetry is preserved in a manifest way.

In this paper, we perform calculations of the trace anomaly by employing the $\mathcal{N} = 4$ regularization of the $\mathcal{N} = 1$ pure Yang-Mills [8, 18], which is suitable for our purpose. (Regularization by dimensional reduction (DRED) [19, 20] can also maintain manifest supersymmetry.¹ But this scheme is not suitable for the study of the anomaly puzzle, as we explain in section 5.) We will find that the trace anomaly is indeed given by the one-loop exact form, if the operator $F_{\mu\nu}F^{\mu\nu}$ is renormalized in a way which is preferred by supersymmetry and the topological nature of $F_{\mu\nu}\tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

In section 2, we give an indirect derivation of the trace anomaly in general field theories, and clarify the point at which the scheme dependence enters. In section 3 the $\mathcal{N} = 4$ regularization is introduced. In section 4, the trace anomaly formula is applied to the $\mathcal{N} = 4$ regularized version of the $\mathcal{N} = 1$ theory. There we discuss how the scheme dependence becomes important. Finally, conclusions are given in section 5. We also discuss the relation between our work and the previous works [5, 6, 7, 8]. Appendix A contains a review of basic properties of the energy-momentum tensor.

¹We will not care about mathematical inconsistency, violation of supersymmetry, and problems with γ_5 in DRED. See Ref. [21] and references therein for discussions on these things.

2 Trace anomaly formula

In this section, we give an indirect derivation of the trace anomaly (up to a total derivative term) in a general field theory which has a Lagrangian description. We will clarify the assumption which is made in the derivation of the trace anomaly.

Let us consider a theory described by a set of fields φ_I , where I is a label specifying the fields. We assume that the theory is renormalized in some renormalization scheme. We consider a correlation function,

$$\left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle = \int [D\varphi] \exp(iS[\varphi, \lambda_a]) \prod_i \varphi_{I_i}(x_i), \quad (12)$$

where λ_a are parameters (e.g., couplings and masses) of the theory, and S is the action. We assume that the field φ_I has mass dimension D_I and the parameter λ_a has mass dimension d_a . By a simple dimensional analysis, we obtain

$$0 = \left(- \sum_i \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + D_{I_i} \right) + \mu \frac{\partial}{\partial \mu} + \sum_a d_a \lambda_a \frac{\partial}{\partial \lambda_a} \right) \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle, \quad (13)$$

where μ is the renormalization scale. On the other hand, the renormalization group (RG) equation tells us that

$$0 = \left(\mu \frac{\partial}{\partial \mu} + \sum_a \beta_a \frac{\partial}{\partial \lambda_a} + \sum_i \gamma_{I_i} \right) \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle. \quad (14)$$

where γ_I is the anomalous dimension of φ_I , and β_a is the beta function defined by

$$\beta_a = \mu \frac{\partial}{\partial \mu} \lambda_a. \quad (15)$$

We assume that there are no operator mixings among φ_I for simplicity.

Next, by using the Ward-Takahashi identity of the energy-momentum tensor given by Eq. (A.7) of Appendix A, we obtain

$$\left\langle \int d^d y i T_\mu^\mu(y) \prod_i \varphi_{I_i}(x_i) \right\rangle = \sum_i \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + d C_{I_i} \right) \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle. \quad (16)$$

where d is the space-time dimension, and C_I are arbitrary constants representing the ambiguity in the definition of the energy-momentum tensor. (see Appendix A for details).

We have assumed that the integral of the total derivative $\int d^d x \partial_\mu (x_\nu T^{\mu\nu})$ vanishes.² Combining Eqs. (13), (14) and (16), we obtain

$$\left\langle \int d^d y i T_\mu^\mu(y) \prod_i \varphi_{I_i}(x_i) \right\rangle = \left(\sum_i (dC_{I_i} - \Delta_{I_i}) + \sum_a (d_a \lambda_a - \beta_a) \frac{\partial}{\partial \lambda_a} \right) \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle, \quad (17)$$

where we have defined $\Delta_I = D_I + \gamma_I$. We can use Eq. (A.9) to rewrite the first term of the right-hand-side as

$$\left(\sum_i (dC_{I_i} - \Delta_{I_i}) \right) \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle = -i \left\langle \left(\int d^d y \sum_I (dC_I - \Delta_I) N_I(y) \right) \prod_i \varphi_{I_i}(x_i) \right\rangle, \quad (18)$$

where N_I is an operator which vanishes by equations of motion (see Appendix A).

Now we make a crucial assumption. Naively, by using the path integral expression for the correlation function, the derivative with respect to λ_a is given as

$$\frac{\partial}{\partial \lambda_a} \left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle = \left\langle i \int d^d y \frac{\partial \mathcal{L}(y)}{\partial \lambda_a} \prod_i \varphi_{I_i}(x_i) \right\rangle, \quad (19)$$

where \mathcal{L} is the Lagrangian. This is known as the quantum action principle [22, 23]. Assuming that this is really the case, we obtain

$$\begin{aligned} & \left\langle \int d^d y T_\mu^\mu(y) \prod_i \varphi_{I_i}(x_i) \right\rangle = \\ & \left\langle \int d^d y \left(\sum_I (\Delta_I - dC_I) N_I(y) + \sum_a (d_a \lambda_a - \beta_a) \frac{\partial \mathcal{L}(y)}{\partial \lambda_a} \right) \prod_i \varphi_{I_i}(x_i) \right\rangle. \end{aligned} \quad (20)$$

Therefore, we finally obtain the operator equation for the trace anomaly,

$$T_\mu^\mu = \sum_a (d_a \lambda_a - \beta_a) \frac{\partial \mathcal{L}}{\partial \lambda_a} + \sum_I (\Delta_I - dC_I) N_I + \partial_\mu j_V^\mu, \quad (21)$$

where $\partial_\mu j_V^\mu$ is a total derivative term which we do not try to determine. As noted above, N_I vanishes by equations of motion. Therefore, as expected, T_μ^μ vanishes up to the total derivative term in the fixed point of RG flow, $d_a \lambda_a - \beta_a = 0$.

In the above derivation, the use of the quantum action principle (19) is crucially important. The validity of the quantum action principle depends on what regularization or

² We should subtract the “cosmological constant term” in $T^{\mu\nu}$ so that the vacuum expectation value is given by $\langle T^{\mu\nu} \rangle = 0$.

renormalization scheme is used. For example, it is known that Eq. (19) holds when minimal subtraction in dimensional regularization is used to renormalize all the parameters and operators of the theory [23]. Then, Eq. (21) reproduces the result of Refs. [15, 16, 17] for the trace anomaly in QCD-like theories up to a total derivative term.

If Eq. (19) is naively applied in the $\mathcal{N} = 1$ pure-Yang-Mills, we would obtain

$$\frac{\partial}{\partial g^{-2}} \langle (\cdots) \rangle = i \left\langle \left(\int d^4x d^2\theta \frac{1}{4} W^\alpha W_\alpha + \text{h.c.} \right) (\cdots) \right\rangle \quad (22)$$

where (\cdots) represents a chain of operators. If this is the case, the trace anomaly is proportional to $(\beta/g^4)W^\alpha W_\alpha$. In the following sections, we examine in what scheme Eq. (22) is really valid in the $\mathcal{N} = 1$ pure Yang-Mills.

3 $\mathcal{N} = 4$ regularization of $\mathcal{N} = 1$ pure Yang-Mills

We employ the $\mathcal{N} = 4$ regularization of the $\mathcal{N} = 1$ pure Yang-Mills which is discussed in Refs. [8, 18]. In this regularization, we take the $\mathcal{N} = 4$ theory with $\mathcal{N} = 4$ supersymmetry softly broken to $\mathcal{N} = 1$ by the mass terms of the adjoint chiral fields. The Lagrangian is given by ³

$$\begin{aligned} \mathcal{L} = & \int d^2\theta \frac{1}{4g_h^2} W^\alpha W_\alpha + \text{h.c.} + \int d^2\theta d^2\bar{\theta} \sum_{i=1}^3 Z_i \Phi_i^\dagger e^{-2V} \Phi_i \\ & + \int d^2\theta \left(\sqrt{2} f_{ABC} \Phi_1^A \Phi_2^B \Phi_3^C + \sum_{i=1}^3 \frac{1}{2} M_i \Phi_i^2 \right) + \text{h.c.}, \end{aligned} \quad (23)$$

where A, B and C are gauge indices, f_{ABC} is the structure constant of the gauge group, and gauge indices are omitted in other terms. The parameters g_h^2 and Z_i ($i = 1, 2, 3$) are given in terms of the gauge coupling g and the theta angle θ as

$$g_h^{-2} = g^{-2} - \frac{i\theta}{8\pi^2}, \quad (24)$$

$$Z_1 Z_2 Z_3 = \text{Re}(g_h^{-2}), \quad (25)$$

where the second equation comes from the constraint of $\mathcal{N} = 4$ supersymmetry. Following Ref. [18], we have normalized the fields so that the coefficient of the term $f_{ABC} \Phi_1^A \Phi_2^B \Phi_3^C$

³ We follow the notation and convention of Wess and Bagger [24], except that a matter kinetic term is given by $\Phi^\dagger e^{-2V} \Phi$, and a gauge field strength chiral field is given by $W_\alpha = \frac{1}{8} \bar{D}^2 (e^{2V} D_\alpha e^{-2V})$.

does not depend on the gauge coupling g . In this normalization, we can maintain the holomorphy [25] when the holomorphic coupling g_h and the mass terms M_i ($i = 1, 2, 3$) are extended to background chiral superfields.

In the energy region much below the scale M_i , this theory becomes the same as the $\mathcal{N} = 1$ pure Yang-Mills, up to terms which are power suppressed by M_i . However, the existence of the chiral multiplets Φ_i makes the theory UV-finite thanks to the known finiteness of the $\mathcal{N} = 4$ theory.⁴ Therefore, this theory can be regarded as the $\mathcal{N} = 1$ pure Yang-Mills regulated by the adjoint chiral fields with the cutoff scale M_i .

Before closing this section, some technical remarks are in order. Just to make our arguments more concrete, we further regulate the theory by e.g., regularization by dimensional reduction (DRED). One reason that we introduce the further regularization is that we would like the regularization to be consistent with the extension of parameters into background superfields so that we can utilize the power of holomorphy. Strictly speaking, this extension maintains only $\mathcal{N} = 1$ supersymmetry, and a possibility exists that the $\mathcal{N} = 4$ regularization might not be compatible with the extension of couplings into background superfields. However, we believe that there is no such incompatibility. For example, let us suppose that the couplings g_h^{-2} and Z_i (and also M_i) are allowed to have non-vanishing constant F -terms (and D -terms for Z_i) with the constraint (25) imposed as a superfield equation. Then, the violation of $\mathcal{N} = 4$ supersymmetry is only soft, i.e., it is broken only by parameters with positive mass dimensions. Therefore all the divergences in dimensionless quantities are still cancelled, and we need no counterterms for the lowest components of g_h^{-2} and Z_i . Other soft breaking parameters are contained in the same multiplet as g_h^{-2} or Z_i , so it is expected that no counterterms are required also for these soft breaking parameters. This idea may be elegantly realized in the scheme of analytic continuation of parameters into superspace discussed in Ref. [26]. For example, RG equations have been given for soft breaking terms by analytically continuing the RG equations for g_h^{-2} and Z_i into superspace, and hence all the soft breaking terms are manifestly RG invariant if that is the case for the lowest components. This suggests that all the UV divergences are cancelled in the present theory even if the parameters are analytically

⁴It is important that the $\mathcal{N} = 4$ supersymmetry is only softly broken by the mass terms of the adjoint chiral fields, and hence the mass terms do not affect the UV divergences of dimensionless quantities.

continued as in Ref. [26].⁵

An important check of the cancellation of divergences in our regularization framework can be found in Ref. [26]. When g_h^{-2} and Z_i have nontrivial soft terms, a counterterm for the mass of the so-called epsilon scalar in DRED is required. At the one-loop level, this counterterm, when analytically continued into superspace, is given as

$$- \int d^2\theta d^2\bar{\theta} \frac{1}{16\pi^2} \frac{1}{\epsilon} \left[t_{Ad} \log(\text{Re}(g_h^{-2})) - \sum_i t_i \log Z_i \right] (\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}), \quad (26)$$

where $\hat{\mu}$ is the Lorentz index of the “compactified” 2ϵ dimensions in DRED, t_{Ad} and t_i are Dinkin indices for the adjoint representation and the representation of the matter field i respectively, and Γ_μ is defined as

$$\Gamma_\mu = \frac{1}{4} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \bar{D}_{\dot{\alpha}} (e^{2V} D_\alpha e^{-2V}). \quad (27)$$

One can check that $\Gamma_{\hat{\mu}}$ is gauge covariant, and hence the above term is allowed by gauge invariance [27]. If Eq. (26) were present, this would give a contribution to the gauge kinetic term by using $\bar{D}^2(\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}) = 4\epsilon W^\alpha W_\alpha$. In our theory, we have $t_i = t_{Ad}$ ($i = 1, 2, 3$), so Eq. (26) vanishes thanks to the condition (25). For this cancellation to happen, it is important that the adjoint chiral fields are normalized in the way which preserves the holomorphy of the superpotential [18], leading to Eq. (25). We expect that this cancellation continues to happen at higher order level. In the following, we assume that the $\mathcal{N} = 4$ regularization is compatible with the extension of the parameters to $\mathcal{N} = 1$ superfields (at least if they are space-time constants, i.e., $\partial_\mu g_h^{-2} = 0$ etc.), and the usual argument based on holomorphy is justified in our regularization framework.⁶

Another reason we regularize the $\mathcal{N} = 4$ theory by e.g. DRED is that we need to treat composite operators. Even in $\mathcal{N} = 4$ theories, composite operators require regularization and renormalization in general. However, fortunately, this problem is also not essential for our purpose of computing the trace anomaly. In our analysis, the only composite

⁵ Here we have introduced the framework of Ref. [26] not because we want to have soft breaking parameters in our theory, but because we want to make the statements about holomorphy more explicit. For our purposes, it is important that we are working in a well-defined (i.e., explicitly regularized) setting where the argument of holomorphy can (at least in principle) be made rigorous.

⁶ Much easier way to ensure finiteness may be to use $\mathcal{N} = 2$ supersymmetry, by taking $Z_1 = \text{Re}(g_h^{-2})$ and $Z_2 = Z_3 = 1$. In this case, we can extend g_h^{-2} to a background $\mathcal{N} = 2$ vector multiplet [28]. Then the absence of radiative corrections is established.

operators which we treat are the energy-momentum tensor and the operators appearing in the trace anomaly. The energy-momentum tensor is finite (at least if improvement [29] is done appropriately). On the other hand, we need the operator $\partial\mathcal{L}/\partial g^{-2}$ for the trace anomaly. For simplicity, let us neglect the mass terms for the adjoint chiral fields. Then this operator is indeed finite in $\mathcal{N} = 4$ theories. The most easy way to see this is to note that the coupling g is exactly marginal, i.e., the coupling does not run in the RG flow for all the possible values of it. Then the operator $\partial\mathcal{L}/\partial g^{-2}$ should have scaling dimension 4 (at least up to possible total derivative terms which we do not care), which coincides with its mass dimension. Therefore there is no anomalous dimension for this operator. This fact indicates that no renormalization is required for this operator.

4 The trace anomaly in $\mathcal{N} = 1$ pure Yang-Mills

Now we discuss the calculations of the beta functions and the trace anomaly in the $\mathcal{N} = 1$ pure Yang-Mills. Let us first discuss the beta functions. In the previous section, we have argued that the manifest holomorphy is expected to be maintained in the regularization process. Then, the low energy physics depends only on the combination

$$\frac{8\pi^2}{g_h^2} - \sum_{i=1}^3 t_{Ad} \log M_i. \quad (28)$$

This statement can be confirmed directly at the one-loop level (by decoupling the adjoint chiral fields, perhaps in the manifestly supersymmetric framework of Ref. [26]). Higher order corrections are absent due to the holomorphy and the dependence of g_h^{-2} on the theta angle.⁷

We introduce a single cutoff scale M (which is taken to be a real positive number) and will give the masses M_i in terms of M . We have infinitely many choices for doing this [18],

⁷This argument for the one-loop exactness becomes invalid when we try to apply it for SQCD regularized by finite $\mathcal{N} = 2$ theories. In the low-energy theory, we have to regard matter wave-function renormalization as an independent parameter in addition to the holomorphic coupling, as is evident in the analyses of e.g. Ref. [26]. We have two (or more) parameters in the low-energy theory, but there is only one gauge coupling g in the high-energy theory. Then it is not possible to choose g so as to fix both the holomorphic gauge coupling and the wave-function renormalization of the low-energy theory when the cutoff scale is changed. This difficulty can be avoided by rescaling the matter fields so that the wave function renormalization of the low energy theory becomes unity. However, this process violates the holomorphy, and hence the one-loop exactness is lost.

and this choice determines the renormalization prescription for the bare coupling g . We will study the RG equation for g which makes the combination (28) invariant when the cutoff scale M is changed. Although we discuss the RG equation for the bare quantity, it is also easy to define renormalized couplings in our regularization framework. We will comment on this point later.

Two particular choices for M_i are often discussed. One choice is given by $M_i = M$, which maintains the holomorphy about the cutoff M . In this case, by requiring that the combination (28) is invariant under the change of M , we obtain

$$M \frac{\partial}{\partial M} g^2 = -\frac{3t_{Ad}g^4}{8\pi^2}. \quad (29)$$

Thus we obtain the one-loop exact beta function in this scheme. The other choice is that we take $M_i = Z_i M$, so that the cutoff M corresponds to the tree level masses of Φ_i when these fields are canonically normalized. Then, the real part of Eq. (28) becomes

$$\frac{8\pi^2}{g^2} - \sum_{i=1}^3 t_{Ad} \log(Z_i M) = \frac{8\pi^2}{g^2} - t_{Ad} \log g^{-2} - 3t_{Ad} \log(M) \quad (30)$$

where Eq. (25) has been used. This choice leads to the NSVZ beta function

$$M \frac{\partial}{\partial M} g^2 = -\frac{1}{8\pi^2} \frac{3t_{Ad}g^4}{1 - t_{Ad}g^2/8\pi^2}. \quad (31)$$

Now we would like to discuss the trace anomaly. Before doing that, a remark on the operator $W^\alpha W_\alpha$ is necessary. The trace anomaly is given in terms of the composite operator $F_{\mu\nu} F^{\mu\nu}$, which is contained in $W^\alpha W_\alpha$. This superfield also contains the operator $F_{\mu\nu} \tilde{F}^{\mu\nu}$. Due to its topological property, the operator $F_{\mu\nu} \tilde{F}^{\mu\nu}$ should not receive multiplicative renormalization, and hence the operator $W^\alpha W_\alpha$ also should have a preferred overall scale. (However, see section 5 for discussion on trouble in DRED if the $\mathcal{N} = 4$ regularization is not used.) This does not mean that the operator $F_{\mu\nu} \tilde{F}^{\mu\nu}$ requires no renormalization at all [30, 31]. Although the overall scale of $F_{\mu\nu} \tilde{F}^{\mu\nu}$ should be determined by its topological nature, radiative corrections of the form $F_{\mu\nu} \tilde{F}^{\mu\nu} \rightarrow F_{\mu\nu} \tilde{F}^{\mu\nu} + \partial_\mu j^\mu$ is not restricted by topological arguments, where j_μ is a gauge invariant operator. In the language of differential forms, we can add an exact 4-form $d(*j)$ to the closed 4-form $F \wedge F$ without affecting it as an element of the de Rham cohomology group. For example, in the case of QED, the

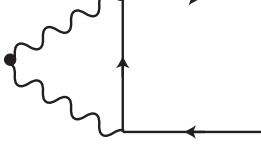


Figure 1: An example of the divergence of the operator $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Wavy lines are gauge fields, and solid lines are fermions. The black filled circle is the operator $F_{\mu\nu}\tilde{F}^{\mu\nu}$.

one-loop diagram in Figure 1 gives a divergence and hence the operator $F_{\mu\nu}\tilde{F}^{\mu\nu}$ needs a counterterm given by

$$[F_{\mu\nu}\tilde{F}^{\mu\nu}] = F_{\mu\nu}\tilde{F}^{\mu\nu} + \left(\frac{3e^4\mu^{2\epsilon}}{8\pi^2} \frac{1}{\epsilon} + c_{\text{finite}} \right) \partial_\mu j_5^\mu, \quad (32)$$

where we have used dimensional regularization,⁸ $j_5^\mu = \frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma_5]\psi$ is the axial current, and $[F_{\mu\nu}\tilde{F}^{\mu\nu}]$ is the renormalized operator. The term c_{finite} is a finite counterterm whose value depends on renormalization prescriptions, and hence the operator $[F_{\mu\nu}\tilde{F}^{\mu\nu}]$ has an ambiguity of the form $c_{\text{finite}}\partial_\mu j_5^\mu$.

One might think that the distinction between $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\partial_\mu j_5^\mu$ is meaningless (in massless limit of fermions) because the axial anomaly equates them. If that were the case, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ would get multiplicative renormalizations. However, the operator $\partial_\mu j_5^\mu$ is equivalent to $F_{\mu\nu}\tilde{F}^{\mu\nu}$ only if we use equations of motion, i.e., $\partial_\mu j_5^\mu \approx F_{\mu\nu}\tilde{F}^{\mu\nu}/8\pi^2$, and they differ by the presence of contact terms when they are inserted into correlation functions. The contact terms are essential in the discussion of the topological properties of $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Meaning of the Adler-Bardeen theorem also requires careful considerations. See Ref. [11] for discussions on the importance of this observation in solving the anomaly puzzle.

In supersymmetric theories, the radiative corrections and counterterms discussed above are extended to superspace as

$$[W^\alpha W_\alpha] = W^\alpha W_\alpha + \bar{D}^2 J, \quad (33)$$

where J is a real vector superfield. Therefore this type of corrections should be allowed in the definition of the composite operator $W^\alpha W_\alpha$.

⁸Here we have avoided DRED because DRED is extremely subtle to perform this type of calculations, because of the relation $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 0$ (see e.g. Ref [21]). We hope that this γ_5 problem does not cause trouble as long as we are working in superspace without looking components of superfields.

Let us calculate the trace anomaly using the formula (21). In the previous section, we have regularized the theory further by DRED for concreteness. It is known that the quantum action principle (19) holds for bare quantities in DRED [21]. Thus we can use Eq. (19) in our regularization. The important point is that we have to include regulator contributions which come from terms containing Φ_i .

If we use the scheme $M_i = M$ which have led to the one-loop exact beta function, we obtain ⁹

$$\frac{\partial \mathcal{L}}{\partial g^{-2}} = \int d^2\theta \frac{1}{4} (W^\alpha W_\alpha + \bar{D}^2 K) + \text{h.c.} \quad (34)$$

$$K = -\frac{1}{2} \sum_{i=1}^3 \frac{\partial Z_i}{\partial g^{-2}} \Phi_i^\dagger e^{-2V} \Phi_i. \quad (35)$$

As explained in the previous section, the operator $\partial \mathcal{L} / \partial g^{-2}$ is finite due to $\mathcal{N} = 4$ supersymmetry. However, the operator $W^\alpha W_\alpha$ itself receives divergent corrections even in $\mathcal{N} = 4$ theories from diagrams such as Figure 1. Thus the term $\bar{D}^2 K$ can be regarded as a regulator term which makes the operator $W^\alpha W_\alpha$ finite. We define the renormalized operator as ¹⁰

$$[W^\alpha W_\alpha]_R = W^\alpha W_\alpha + \bar{D}^2 K. \quad (36)$$

As explained above, the addition of $\bar{D}^2 K$ does not violate the topological nature of $F_{\mu\nu} \tilde{F}^{\mu\nu}$, and hence it should be allowed. Therefore, the trace anomaly is given as

$$\begin{aligned} T_\mu^\mu &\approx -\frac{\partial g^{-2}}{\partial M} \frac{\partial \mathcal{L}}{\partial g^{-2}} + \partial_\mu j_V^\mu \\ &= -\frac{3t_{Ad}}{32\pi^2} \int d^2\theta [W^\alpha W_\alpha]_R + \text{h.c.} + \partial_\mu j_V^\mu \end{aligned} \quad (37)$$

where \approx means that we have used the equations of motion, $N_I \approx 0$. Therefore, we have determined that the coefficient of the operator $[W^\alpha W_\alpha]_R$ in the trace anomaly is one-loop exact.

⁹ We also have to include contributions from gauge fixing terms and ghosts. However, we expect that these contributions are BRST exact, as in the case of QCD [15, 16, 17]. We neglect them in this paper.

¹⁰ The operator K , and hence $[W^\alpha W_\alpha]_R$, may look to be depending on the precise choice of Z_i , but it is not so. If we use different wave function renormalizations Z'_i satisfying Eq. (25) and obtain K' , then the difference $K' - K$ is a current of $\text{SU}(3) \subset \text{SU}(4)_R$ symmetry of the $\mathcal{N} = 4$ theory. In particular, it is UV-finite. Then the argument of the heavy field decoupling theorem (see e.g. Ref. [32]) may tell us that $K' - K$ is power suppressed by the heavy field masses M_i . The UV-finiteness is essential, because otherwise momentum integrals such as $\int d^4k (k^2 + M^2)^{-2}$ spoil the suppression.

Next, let us perform the calculation in the scheme $M_i = Z_i M$, which have led to the NSVZ beta function. Then, we obtain

$$\frac{\partial \mathcal{L}}{\partial g^{-2}} = \int d^2\theta \left(\frac{1}{4} [W^\alpha W_\alpha]_R + \frac{1}{2} \sum_{i=1}^3 \frac{\partial \log Z_i}{\partial g^{-2}} M_i \Phi_i^2 \right) + \text{h.c.} \quad (38)$$

In this case, we have additional contributions coming from the regulator mass terms. Therefore, the naive use of Eq. (22) is invalid if the regulator contribution is not taken into account seriously. In the low energy limit (i.e., in the energy region much below M), the operator $M_i \Phi_i^2$ becomes

$$M_i \Phi_i^2 \rightarrow -\frac{t_{Ad}}{16\pi^2} [W^\alpha W_\alpha]_R \quad (i = 1, 2, 3). \quad (39)$$

At the one-loop level, this equation can be checked directly. The holomorphy restricts higher order corrections to the form $W^\alpha W_\alpha \rightarrow W^\alpha W_\alpha + \bar{D}^2 J$. We have assumed that these corrections are such that the operator $W^\alpha W_\alpha$ is replaced by $[W^\alpha W_\alpha]_R$. This is because the operator Φ_i^2 is UV-finite since it is protected by $\mathcal{N} = 4$ superconformal symmetry.

Using Eq. (39), we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial g^{-2}} &\rightarrow \int d^2\theta \frac{1}{4} \left(1 - \frac{t_{Ad}}{8\pi^2} \sum_{i=1}^3 \frac{\partial \log Z_i}{\partial g^{-2}} \right) [W^\alpha W_\alpha]_R + \text{h.c.} \\ &= \int d^2\theta \frac{1}{4} \left(1 - \frac{t_{Ad} g^2}{8\pi^2} \right) [W^\alpha W_\alpha]_R + \text{h.c.}, \end{aligned} \quad (40)$$

where we have used Eq. (25). Combining this result with the NSVZ beta function (31), we again get the same answer given by Eq. (37). Therefore, the one-loop exact form of the trace anomaly is also confirmed in this scheme.

Finally, let us comment on renormalization in the $\mathcal{N} = 4$ regularization. We have treated bare couplings up to now, but it is very easy to define a renormalized coupling since we know that the low energy physics depends only on the combination (28). For example, we can define a renormalized coupling g_R as

$$g_R^{-2}(\mu) = g^{-2} - \sum_{i=1}^3 \frac{t_{Ad}}{8\pi^2} \log \left(\frac{M_i}{\mu} \right), \quad (41)$$

where μ is the renormalization scale. Now we can let the cutoff M_i be infinity at the end of the calculations with g_R fixed. The derivation of the trace anomaly is completely parallel

to the one given above. Although it is not so evident that the operator $[W^\alpha W_\alpha]_R$ remains finite in the limit $M_i \rightarrow \infty$, we expect that this is the case since the energy-momentum tensor (and hence its trace) is UV-finite.

5 Conclusions and comparison with the literature

In this paper we have investigated the trace anomaly in $\mathcal{N} = 1$ pure Yang-Mills to solve the anomaly puzzle in this theory. The trace anomaly is often quoted as $-(\beta/4g^4)F_{\mu\nu}F^{\mu\nu}$, but the validity of this formula depends on whether the quantum action principle given by Eq. (22) (or more generally Eq. (19)) is really true or not. To settle this issue, we have explicitly regularized the theory by using the $\mathcal{N} = 4$ regularization. Then the regulator contribution to Eq. (22) can be explicitly seen, and this contribution depends on what scheme we use. In the scheme in which we obtain the one-loop exact beta function, it is almost correct to use Eq. (22) naively with the regulator contribution neglected. The regulator contribution only plays a role of making the operator $W^\alpha W_\alpha$ renormalized, $[W^\alpha W_\alpha]_R = W^\alpha W_\alpha + \bar{D}^2 K$, without spoiling the topological nature of $F_{\mu\nu}\tilde{F}^{\mu\nu}$ contained in $W^\alpha W_\alpha$. However, if the scheme is used in which we obtain the NSVZ beta function, the regulator contribution is not at all negligible, as seen in Eq. (40). If this contribution is correctly taken into account, we obtain the same answer for the trace anomaly in both of the schemes. The result for the trace anomaly has the one-loop exact form, and hence is consistent with the supercurrent equation and the Adler-Bardeen theorem.

Although the $\mathcal{N} = 4$ regularization discussed in this paper is applicable only to the $\mathcal{N} = 1$ pure Yang-Mills, we strongly believe that there exists a renormalization scheme in which the supercurrent equations of Refs. [7, 11] which include matter fields are really justified. This “desirable renormalization scheme” should also justify various exact results in supersymmetric gauge theories. One of the properties which the desirable scheme should possess is the quantum action principle (19), at least if we use the couplings which can be extended to background superfields.

In the rest of this section, we would like to comment on how our result is consistent with the results obtained in Refs. [5, 6, 7, 8]. Grisaru, Milewski and Zanon [5, 6] studied the supercurrent equation by employing DRED. They constructed two supercurrents which we

will call $\mathcal{J}_{\alpha\dot{\alpha}}^{(1)}$ and $\mathcal{J}_{\alpha\dot{\alpha}}^{(2)}$, and performed calculations at the two-loop level. One supercurrent $\mathcal{J}_{\alpha\dot{\alpha}}^{(1)}$ satisfies the equation (in our notation and convention)

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}}^{(1)} \approx D_{\alpha} \left(\frac{1}{3} \frac{\beta_{\text{two-loop}}}{g^4} [W^{\alpha} W_{\alpha}]_{\text{DR}} \right), \quad (42)$$

where the coupling g and the operator $[W^{\alpha} W_{\alpha}]_{\text{DR}}$ are renormalized by minimal subtraction in DRED (i.e., the DR or $\overline{\text{DR}}$ scheme), and $\beta_{\text{two-loop}}$ is the two-loop beta function which coincides with the NSVZ beta function at the two-loop level. The other supercurrent $\mathcal{J}_{\alpha\dot{\alpha}}^{(2)}$ satisfies

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}}^{(2)} \approx D_{\alpha} \left(\frac{1}{3} \frac{\beta_{\text{one-loop}}}{g^4} [W^{\alpha} W_{\alpha}]_{\text{DR}} \right) + b \bar{D}^2 D_{\alpha} [\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}]_{\text{DR}}, \quad (43)$$

where $\beta_{\text{one-loop}}$ is the one-loop beta function, and b is a coefficient which they did not explicitly compute. They claimed that $\mathcal{J}_{\alpha\dot{\alpha}}^{(1)}$ contains the energy-momentum tensor, while the lowest component of $\mathcal{J}_{\alpha\dot{\alpha}}^{(2)}$ satisfies the Adler-Bardeen theorem.

Let us first consider Eq. (42). At first sight, this equation may look inconsistent with our result, because it seems to give the trace anomaly which is not one-loop exact. However, this is not necessarily the case. The problem is that the normalization of the operator $W^{\alpha} W_{\alpha}$ is quite ambiguous in DRED. We have discussed in section 4 that this operator has an ambiguity of adding a term $\bar{D}^2 J$, where J is a gauge invariant real vector superfield. However, in DRED, there exists the operator $\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}$ (see section 3) satisfying

$$\bar{D}^2 (\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}) = 4\epsilon W^{\alpha} W_{\alpha}. \quad (44)$$

By choosing $J \propto (\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}})$, the normalization of the operator $W^{\alpha} W_{\alpha}$ becomes completely ambiguous. This ambiguity may be related to the fact that there is no instanton in $4 - 2\epsilon$ dimensions, and hence the topological arguments do not work in this scheme.

More concretely, suppose that the divergences of $W^{\alpha} W_{\alpha}$ are of the form $(1/\epsilon^n) \bar{D}^2 (\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}})$ ($n = 1, 2, \dots$) [27], and we define a renormalized operator $[W^{\alpha} W_{\alpha}]$ as

$$\begin{aligned} [W^{\alpha} W_{\alpha}] &= W^{\alpha} W_{\alpha} + \sum_{n=1}^{\infty} \frac{c_n(g)}{\epsilon^n} \frac{1}{4} \bar{D}^2 (\Gamma^{\hat{\mu}} \Gamma_{\hat{\mu}}) \\ &= (1 + c_1(g)) W^{\alpha} W_{\alpha} + \sum_{n=2}^{\infty} \frac{c_n(g)}{\epsilon^{n-1}} W^{\alpha} W_{\alpha}. \end{aligned} \quad (45)$$

The $n = 1$ term eventually gives a finite counterterm. Then, we have a choice for the renormalization prescription. One choice is to set $c_1(g) = 0$, which corresponds to the DR scheme. But we can also choose a scheme in which $c_1(g) \neq 0$. The freedom in choosing $c_1(g)$ leads to the ambiguity of the renormalized operator $[W^\alpha W_\alpha]$.

One may consider that it is meaningless to say that the divergences are of the form $(1/\epsilon^n)\bar{D}^2(\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}})$ instead of $(1/\epsilon^n)W^\alpha W_\alpha$, since we have the relation (44). In fact, this statement is quite nontrivial. The divergent pole $(1/\epsilon^n)\bar{D}^2(\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}})$ arises from n or higher loop diagrams, and hence the coefficient $c_n(g)$ is of the form $c_n(g) \sim g^{2n} + \dots$. This means that the divergent pole $(1/\epsilon^n)W^\alpha W_\alpha$ comes from $n + 1$ or higher loops and has a coefficient of order $g^{2(n+1)}$. This is suppressed by additional g^2 than naively expected. In fact, this suppression can be confirmed by the two-loop calculation of Grisaru, Milewski and Zanon [5, 6]. In our notation and convention, their result is summarized as ¹¹

$$[W^\alpha W_\alpha]_{\text{DR}} = \left(1 - \frac{3}{\epsilon} \left(\frac{t_{Ad}g^2}{16\pi^2}\right)^2 + \mathcal{O}(g^6)\right) W^\alpha W_\alpha. \quad (46)$$

Having confidence that the divergence is indeed $(1/\epsilon^n)\bar{D}^2(\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}})$, we suspect that there exists a “hidden divergence” of the form $(1/\epsilon)\bar{D}^2(\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}})$, which should be subtracted by nonzero $c_1(g)$. (In fact, Eq. (26) is a similar kind of such hidden divergence which becomes clear by the analytic continuation of parameters into superspace [26].) This subtraction will give a different normalization of the renormalized operator $[W^\alpha W_\alpha]$ than the operator $[W^\alpha W_\alpha]_{\text{DR}}$ subtracted by DR.

The $\mathcal{N} = 4$ regularization avoids the above difficulty by effectively replacing the counterterms as

$$\sum_{n=1}^{\infty} \frac{c_n(g)}{\epsilon^n} \frac{1}{4} \bar{D}^2(\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}}) \rightarrow \bar{D}^2 K, \quad (47)$$

where K is defined in Eq. (35). In this way, the structure of the divergence becomes much more clear. We can uniquely specify the overall normalization of $[W^\alpha W_\alpha]$, which is preferred by the topological nature of $F_{\mu\nu}\tilde{F}^{\mu\nu}$. (Strictly speaking, we have regularized the

¹¹ One should note that what they called the bare operator $W_\alpha^{(0)}$ is in fact $(g/g_0)W_\alpha$ in our notation, where g_0 is the bare gauge coupling in DRED. The operator W_α in their notation coincides with our W_α because of the relation $Z_V = (g/g_0)^2$, where Z_V is defined in their paper. The supercurrent should have been defined as $(1/g_0^2)W_\alpha\bar{W}_\alpha + \dots$ instead of $W_\alpha^{(0)}\bar{W}_\alpha^{(0)} + \dots$.

$\mathcal{N} = 4$ theory by DRED to make our arguments concrete. But we believe that this is not essential. Any other regularization is possible as long as the arguments of section 4 make sense.) Our renormalization prescription for $W^\alpha W_\alpha$ gives the one-loop exact coefficient in the trace anomaly.

As to the second supercurrent $\mathcal{J}_{\alpha\dot{\alpha}}^{(2)}$, our guess is that this supercurrent is simply related to the first one by

$$\mathcal{J}_{\alpha\dot{\alpha}}^{(2)} = \frac{\beta_{\text{one-loop}}}{\beta_{\text{two-loop}}} \mathcal{J}_{\alpha\dot{\alpha}}^{(1)} + [\text{evanescent}], \quad (48)$$

where by the term [evanescent], we mean an operator which vanishes in the limit $\epsilon \rightarrow 0$ when it is inserted into a renormalized correlation function. In dimensional regularization or reduction, it often happens that two different-looking operators are in fact the same up to evanescent terms (see e.g., Ref. [32]). In fact, in the construction of the two supercurrents, operators such as $[D_\alpha, \bar{D}_{\dot{\alpha}}][\Gamma^{\hat{\mu}}\Gamma_{\hat{\mu}}]_{\text{DR}}$ are necessary which seem to be evanescent operators. We do not perform a detailed study on this point, but the existence of the two different supercurrents cannot be confirmed unless the possibility (48) is excluded. (However, the idea of two supercurrents becomes important when matter fields are included in the theory [11, 12, 33].)

Next we discuss the result obtained by Arkani-Hamed and Murayama [8]. They computed the anomalies in dilatation transformations. Their result can be summarized as follows. If we take the dilatation transformation of the gauge superfield $V(x, \theta, \bar{\theta})$ as

$$V(x, \theta, \bar{\theta}) \rightarrow V(e^t x, e^{t/2} \theta, e^{t/2} \bar{\theta}), \quad (49)$$

where t is the parameter of the transformation, we obtain the anomaly given by

$$\int d^4x \int d^2\theta \left(t \frac{3t_{Ad}}{32\pi^2} \right) W^\alpha W_\alpha + \text{h.c.} \quad (50)$$

On the other hand, we can canonically normalize the gauge field as $V = g_c V_c$, where g_c is the “canonical gauge coupling”, and perform the dilatation transformation as

$$g_c V_c(x, \theta, \bar{\theta}) \rightarrow g'_c(t) V_c(e^t x, e^{t/2} \theta, e^{t/2} \bar{\theta}), \quad (51)$$

where $g'_c(t)$ is chosen so that the gauge field is canonically normalized after the dilatation transformation. Then the anomaly is given by

$$\int d^4x \int d^2\theta \frac{1}{4} \left(t \frac{3t_{Ad}}{8\pi^2} - \frac{t_{Ad}}{8\pi^2} \log \left(\frac{g'_c(t)}{g_c^2} \right) \right) W^\alpha (g'_c(t) V_c) W_\alpha (g'_c(t) V_c) + \text{h.c.} \quad (52)$$

Here $g'_c(t)$ is determined by the equation

$$\frac{1}{g'_c{}^2(t)} = \frac{1}{g_c^2} + t \frac{3t_{Ad}}{8\pi^2} - \frac{t_{Ad}}{8\pi^2} \log \left(\frac{g_c'^2(t)}{g_c^2} \right). \quad (53)$$

Taking t infinitesimal, we get

$$\frac{1}{g'_c{}^2(t)} = \frac{1}{g_c^2} - t \frac{\beta_{\text{NSVZ}}}{g_c^4} \quad (54)$$

where β_{NSVZ} is the NSVZ beta function. In the transformation (51), the classical action also changes. The total change of the path integral $\int [DV] e^{iS}$ in this case is given by

$$\int d^4x \int d^2\theta t \beta_{\text{NSVZ}} \left. \frac{\partial}{\partial g_c^2} \right|_{V_c} \left(\frac{1}{4g_c^2} W^\alpha(g_c V_c) W_\alpha(g_c V_c) \right) + \text{h.c.} \quad (55)$$

Therefore, it was claimed that there are two dilatation anomalies. One is given by Eq. (50), which has the one-loop exact form, while the other is given by Eq. (55) which, at first sight, has higher order corrections. This result may seem to contradict with our result, since we have obtained the unique trace anomaly, up to a total derivative term and the ambiguity of choosing C_I discussed in Appendix A.

However, a closer look at Eq. (55) gives us the resolution to this problem. We can rewrite this equation as

$$t \beta_{\text{NSVZ}} \left. \frac{\partial S}{\partial g_c^2} \right|_{V_c} = t \beta_{\text{NSVZ}} \left(\left. \frac{\partial S}{\partial g_c^2} \right|_V + \int d^4x d^2\theta d^2\bar{\theta} \frac{1}{2g_c^2} \frac{\delta S}{\delta V} V \right), \quad (56)$$

where S is the action and $\delta S/\delta V$ is the functional differentiation of the action with respect to V . Naively, $V(\delta S/\delta V)$ vanishes by equations of motion. However, the computation of the anomaly under the rescaling $V \rightarrow (\text{const.})V$ performed by Arkani-Hamed and Murayama indicates that the equation of motion in their regularization scheme should really be given as (see Appendix A, especially Eqs. (A.8), (A.9) and (A.10)),

$$0 \approx N_V(x, \theta, \bar{\theta}) \equiv \frac{\delta S}{\delta V} V - \frac{t_{Ad}}{16\pi^2} \left(-\frac{1}{2} (e^{2V} D^\alpha e^{-2V}) W_\alpha \right) + \text{h.c.} + (\text{total derivative}), \quad (57)$$

where one should note that the second term satisfies

$$-\frac{1}{4} \bar{D}^2 \left(-\frac{1}{2} (e^{2V} D^\alpha e^{-2V}) W_\alpha \right) = W^\alpha W_\alpha. \quad (58)$$

Then, Eq. (56) becomes

$$\begin{aligned}
& t\beta_{\text{NSVZ}} \int d^4x d^2\theta \left(-\frac{1}{4g_c^4} + \frac{t_{Ad}}{32\pi^2 g_c^2} \right) W^\alpha W_\alpha + \text{h.c.} + \int d^4x d^2\theta d^2\bar{\theta} t \frac{\beta_{\text{NSVZ}}}{2g_c^2} N_V \\
& = \int d^4x \int d^2\theta \left(t \frac{3t_{Ad}}{32\pi^2} \right) W^\alpha W_\alpha + \text{h.c.} + \int d^4x d^2\theta d^2\bar{\theta} t \frac{\beta_{\text{NSVZ}}}{2g_c^2} N_V.
\end{aligned} \tag{59}$$

Therefore, the difference of the two anomalies is only the term containing N_V , which vanishes by the equation of motion. This just reflects the ambiguity of the definition of the energy-momentum tensor discussed in Appendix A. In fact, by setting $\delta x^\mu = x^\mu$ in Eqs. (A.3) and (A.5) of Appendix A, we can see that the two dilatation anomalies computed above are just the trace T_μ^μ with different values of C_I . In this way, we obtain the one-loop exact coefficient for the operator $W^\alpha W_\alpha$ in the dilatation anomaly, which is consistent with our result.

Shifman and Vainshtein [7] gave a supercurrent equation which is consistent with our result on the trace anomaly. Although there is no contradiction in the supercurrent equation, their interpretation of the anomaly puzzle is different from ours. They argued that the one-loop contribution to the beta function comes from UV region, but higher loop effects are from IR region. Since operator equations are given in terms of UV quantities, they claimed that the one-loop beta function should appear in the trace anomaly. However, as emphasized by Arkani-Hamed and Murayama [8], both the one-loop beta function and the NSVZ beta function can be derived by only using the information of UV physics. Indeed, the derivation of the NSVZ beta function we reviewed in section 4 has used only the decoupling of the adjoint chiral fields, Eq. (28), which is completely determined in UV region.¹² Then the question of which beta function should appear in the trace anomaly comes back to us. This is precisely the problem we have studied in this paper. Although there are many renormalization schemes which give different beta functions, only one scheme possesses the desired property

$$\frac{\partial}{\partial g^{-2}} \langle (\cdots) \rangle = i \left\langle \left(\int d^4x d^2\theta \frac{1}{4} [W^\alpha W_\alpha]_R + \text{h.c.} \right) (\cdots) \right\rangle \tag{60}$$

for the appropriately renormalized operator $[W^\alpha W_\alpha]_R$. Without this property, the trace anomaly formula is simply false.

¹²We have not even used the Jacobian from the path-integral measure as in Ref. [8].

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Appendix A Energy-momentum tensor

In this appendix we review basic properties of the energy-momentum tensor. The energy-momentum tensor can be defined as the Noether current of the Poincare symmetry. Let us consider a theory described by fields $\varphi_I(x)$ which are in some representations of the Lorentz group. We treat φ_I as if they are bosons for simplicity, but a generalization to fermions is obvious. An infinitesimal transformation of the coordinates is given as

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu(x). \quad (\text{A.1})$$

We define a transformation of the field φ_I under the above coordinate transformation as

$$\varphi_I(x) \rightarrow \varphi'_I(x) = \varphi_I(x) + \delta\varphi_I(x), \quad (\text{A.2})$$

$$\delta\varphi_I(x) = \delta x^\mu \partial_\mu \varphi_I(x) + \frac{1}{2} \partial_{[\mu} \delta x_{\nu]} \Sigma_I^{\mu\nu} \varphi_I(x) + \partial_\mu \delta x^\mu C_I \varphi_I(x), \quad (\text{A.3})$$

where $\Sigma_I^{\mu\nu}$ is the Lorentz symmetry generator acting on φ_I , and C_I is an arbitrary constant. This definition is taken so that it coincides with Poincare transformation when δx^μ is given as $\delta x^\mu = \omega^\mu{}_\nu x^\nu + a^\mu$, where a_μ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are constants.

Let us consider a correlation function

$$\left\langle \prod_i \varphi_{I_i}(x_i) \right\rangle = \int [D\varphi] e^{iS[\varphi]} \prod_i \varphi_{I_i}(x_i). \quad (\text{A.4})$$

Then, by performing the above transformation of the fields in the path integral, we obtain

$$0 = - \left\langle i \int d^d x T^{\mu\nu} \partial_\mu \delta x_\nu \prod_i \varphi_{I_i}(x_i) \right\rangle + \sum_i \left\langle \delta\varphi_{I_i}(x_i) \prod_{j \neq i} \varphi_{I_j}(x_j) \right\rangle, \quad (\text{A.5})$$

where d is the space-time dimension. This is the defining equation of the energy-momentum tensor, but a slight ambiguity remains. There is a freedom to change $T^{\mu\nu}$ as $T^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$,

where $A^{\rho\mu\nu}$ is an arbitrary tensor satisfying $A^{\rho\mu\nu} = -A^{\mu\rho\nu}$. We can use this freedom to make $T^{\mu\nu}$ symmetric, $T^{\mu\nu} = T^{\nu\mu}$. First, note that if $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ where $\omega_{\mu\nu}$ is an antisymmetric constant tensor, the path integral measure times the factor $e^{iS[\varphi]}$ is invariant by the Lorentz symmetry. Then $\int d^d x T^{[\mu\nu]} \omega_{\mu\nu} = 0$ for an arbitrary constant $\omega_{\mu\nu}$, and hence $T^{[\mu\nu]}$ is a total derivative, $T^{[\mu\nu]} = \partial_\rho B^{\rho\mu\nu}$. We can define a symmetric energy-momentum tensor as $T^{\mu\nu} - \partial_\rho (B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$, i.e., $A^{\rho\mu\nu} = -B^{\rho\mu\nu} + B^{\mu\rho\nu} + B^{\nu\rho\mu}$. In the following, we use this symmetric tensor. Even after making $T^{\mu\nu}$ symmetric, there still remains a freedom to change $T^{\mu\nu}$ as

$$T^{\mu\nu} \rightarrow T^{\mu\nu} + \partial_\rho \partial_\sigma C^{\mu\rho\nu\sigma}, \quad (\text{A.6})$$

where $C_{\mu\rho\nu\sigma}$ is an arbitrary field satisfying $C_{\mu\rho\nu\sigma} = -C_{\rho\mu\nu\sigma} = C_{\nu\sigma\mu\rho}$. This is the so-called improvement of the energy-momentum tensor. If one prefers to define $T^{\mu\nu}$ by functional differentiation of the action with respect to the metric tensor $g_{\mu\nu}$, this freedom corresponds to adding a term $-\frac{1}{4} \int d^d R_{\mu\rho\nu\sigma} C^{\mu\rho\nu\sigma}$ to the action, where $R_{\mu\rho\nu\sigma}$ is the Riemann tensor.

Functionally differentiating Eq. (A.5) by $\delta x^\nu(y)$, we get the Ward-Takahashi identity

$$\begin{aligned} & \left\langle \partial_\mu T^{\mu\nu}(y) \prod_i \varphi_{I_i}(x_i) \right\rangle = \\ & i \sum_i \left\langle \left(\delta^{(d)}(x_i - y) \partial^\nu \varphi_{I_i}(x_i) - \left(\frac{1}{2} \Sigma_{I_i}^{\mu\nu} + C_{I_i} \eta^{\mu\nu} \right) (\partial_y)_\mu \delta^{(d)}(x_i - y) \varphi_{I_i}(x_i) \right) \prod_{j \neq i} \varphi_{I_j}(x_j) \right\rangle. \end{aligned} \quad (\text{A.7})$$

In particular, if $y \neq x_i$, this equation is just the conservation of the energy-momentum tensor, $\partial_\mu T^{\mu\nu} \approx 0$, where \approx means that the equation is valid up to contact terms.

Aside from the improvement, there exists the ambiguity of choosing C_I in the definition of the energy-momentum tensor. To investigate this point, let us consider a rescaling transformation of the field φ_I as

$$\varphi_I(x) \rightarrow (1 + \alpha(x)) \varphi_I(x), \quad (\text{A.8})$$

where $\alpha(x)$ is an infinitesimal function. By performing this rescaling in the correlation function (A.4) and functionally differentiating the result by $\alpha(y)$, we obtain

$$0 = \left\langle i N_I(y) \prod_i \varphi_{I_i}(x_i) \right\rangle + \sum_{I_i=I} \left\langle \delta^{(d)}(x_i - y) \varphi_{I_i}(x_i) \prod_{j \neq i} \varphi_{I_j}(x_j) \right\rangle. \quad (\text{A.9})$$

Here we have defined the operator N_I as

$$N_I(x) = \frac{\delta S[\varphi]}{\delta \varphi_I(x)} \varphi_I(x) + A_I(x), \quad (\text{A.10})$$

where $A_I(x)$ is the anomaly arising from the path integral measure in the transformation (A.8). The operator N_I vanishes by equations of motion, $N_I \approx 0$, since there are only contact terms in Eq. (A.9) aside from the term containing N_I . Using N_I , one can see that the ambiguity in the definition of the energy-momentum tensor represented by C_I is of the form

$$T^{\mu\nu} = T^{\mu\nu}|_{C_I=0} - \eta^{\mu\nu} \sum_I C_I N_I. \quad (\text{A.11})$$

Thus, energy-momentum tensors with different values of C_I are the same up to the equations of motion, $N_I \approx 0$. More generally, any symmetric tensor which vanishes by equations of motion can be added to $T^{\mu\nu}$ without violating the conservation equation $\partial_\mu T^{\mu\nu} \approx 0$. One can also check that this ambiguity does not affect the generators of the Poincare symmetry $P^\mu = \int d^{d-1}x T^{0\mu}$ and $J^{\mu\nu} = \int d^{d-1}x (x^\mu T^{0\nu} - x^\nu T^{0\mu})$ when Eq. (A.7) is integrated to give the commutation relations $[P_\mu, \varphi_I] = i\partial_\mu \varphi_I$ and $[J_{\mu\nu}, \varphi_I] = i(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{I\mu\nu})\varphi_I$.

Finally, let us comment on the role of C_I at the fixed point of RG flow. In section 2, we derive the formula for the trace of the energy-momentum tensor given by Eq. (21). By using that formula, the dilatation current

$$j_D^\mu = x_\nu T^{\mu\nu} - j_V^\mu \quad (\text{A.12})$$

can be shown to satisfy the Ward-Takahashi identity at the fixed point $d_a \lambda_a - \beta_a = 0$,

$$\begin{aligned} & \left\langle \partial_\mu j_D^\mu(y) \prod_i \varphi_{I_i}(x_i) \right\rangle = \\ & i \sum_i \left\langle \delta^{(d)}(x_i - y) (x_i^\nu \partial_\nu \varphi_{I_i}(x_i) + \Delta_{I_i} \varphi_{I_i}(x_i)) \prod_{j \neq i} \varphi_{I_j}(x_j) \right\rangle \\ & + i(\partial_y)_\mu \sum_i \left\langle \delta^{(d)}(x_i - y) \left(-\frac{1}{2} y_\nu \Sigma_{I_i}^{\mu\nu} - y^\mu C_{I_i} \right) \varphi_{I_i}(x_i) \prod_{j \neq i} \varphi_{I_j}(x_j) \right\rangle. \quad (\text{A.13}) \end{aligned}$$

If we integrate this equation over y^μ to obtain the integrated version of the Ward-Takahashi identity, the final line of Eq. (A.13) does not contribute at all. Thus the arbitrary parameter C_I has no physical meaning in the scaling symmetry.

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